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A CONSISTENT THEORY OF GEOMETRICALLY NON-LINEAR SHELLS WITH AN INDEPENDENT ROTATION VECTOR

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Abstract—For shells undergoing finite rotations, a general theory is formulated in terms of consistent displacement and force variables. An independent rotation vector is used for the description of the deformation state. The strain–displacement equations are obtained considering shear deformations. These relations are then transformed by a variational procedure into consistent equilibrium equations and boundary conditions, the validity of which is also confirmed by an independent two-dimensional derivation. The paper closes with the physical interpretation of the force variables and the formulation of the constitutive equations.

1. INTRODUCTION

Every new attempt at deriving a non-linear shell theory has, due to the considerable development of computational mechanics, to be oriented to new aspects. In their general form, the non-linear theories are not in themselves accessible to numerical analysis and have to be transformed for this purpose into incremental formulations which are, however, always linear in the unknown variables, independently of the order of the nonlinearity of the initial equations[1–4]. Thus, the new theories need not be simplified by assumptions concerning the order of magnitude of the deformation variables. For numerical analysis, it is more suitable to have theories with a wide range of applicability than simplified ones the validity of which could hardly be checked by a given practical problem. Moreover, it should be remembered that the deformation variables (displacements, rotations, strains, deformation gradients,...) are related by geometrical constraints so that the order of magnitude of certain variables cannot be judged independently of the others. If we allow finite rotations[5, 6] for instance, this will have, according to the well-known theorems of Gauß and Codazzi[1, 7], consequences for the variables connected with the first fundamental form of the deformed middle surface.

A further question arising with regard to the derivation of a shell theory is the choice of a suitable kinematic assumption about the transformation of the three-dimensional initial equations into two-dimensional ones. The essential advantage of the Kirchhoff–Love theories, namely the description of the deformation by only three independent displacement components, is not so essential from a purely numerical point of view[2, 3]. It should also be mentioned that the structure of the corresponding equations, especially of the dynamic boundary conditions[8] are too complicated even for numerical analysis. On the other hand kinematic assumptions allowing shear deformations ensure a very systematical derivation, leading to equations with a relatively simple structure, as can be observed in the present derivation. Furthermore, shear deformation theories can be regarded as a suitable starting point for the derivation of Kirchhoff–Love theories. Finally, it should be mentioned that theories with shear deformations have been used successfully in recent years for numerical applications[2, 4].

From the mechanical point of view[1, 9, 10], each shell theory has to satisfy the requirements of consistency. A consistent theory will be understood here to be a formulation the equilibrium and dynamic boundary conditions of which are of the same order of accuracy as the kinematic relations involving the adopted kinematic model. This requirement can be fulfilled by a three-dimensional variational derivation leading to equations which are related by means of Green's theorem. The use of the principle of virtual work for this purpose, instead of the principle of stationary potential energy[8], has a considerable advantage: it allows the relationship existing between the equations in question to be discovered independent of the material behaviour. Thus, the mathematical rules resulting from this relationship for the corresponding operators[11] can be used in an elegant manner for the derivation of non-linear theories.

In variational derivations, the definition of force variables is obtained formally by a mathematical procedure. For practical applications, it is therefore necessary to relate them to physical variables which can be interpreted physically in a two-dimensional shell element. This means, that a variational derivation ensuring a consistent formulation has to be complemented by a further two-dimensional investigation in order to satisfy the requirements of practical applications.

According to the above discussion, the purpose of this paper can be formulated as follows. It consists of a derivation of a non-linear theory considering shear deformations. Thereby, assumptions concerning the order of magnitude of deformations will be avoided in order not to restrict the applicability of the theory. The basis of the derivation is the principle of virtual work of a three-dimensional continuum. Internal and boundary forces are first introduced by a variational procedure and are then related to those defined on the two-dimensional shell element. It is shown that the equilibrium conditions first derived variationally can also be obtained by a purely two-dimensional derivation.

2. GEOMETRICAL RELATIONS

Let $\mathring{\mathbf{r}} = \mathring{\mathbf{r}}(\theta^\alpha)$ be the position vector of a point $\mathring{P}$ of the undeformed middle surface $\mathring{F}$ where θ^α are curvilinear coordinates. All the geometrical variables associated with the middle surface will be denoted by the usual notations[1, 7, 12], having however the suffix (°) if they refer to undeformed state. Thus

$$\text{base vectors:}\quad \mathring{\mathbf{a}}_\alpha = \mathring{\mathbf{r}}_{,\alpha},\quad \mathring{\mathbf{a}}^\alpha = \mathring{a}^{\alpha\beta}\mathring{\mathbf{a}}_\beta \tag{1}$$

$$\text{metric tensors:}\quad \mathring{a}_{\alpha\beta} = \mathring{\mathbf{a}}_\alpha\cdot\mathring{\mathbf{a}}_\beta,\quad \mathring{a}^{\alpha\beta} = \mathring{\mathbf{a}}^\alpha\cdot\mathring{\mathbf{a}}^\beta \tag{2}$$

$$\text{determinant:}\quad \mathring{a} = \mathring{a}_{11}\mathring{a}_{22}-(\mathring{a}_{12})^2 \tag{3}$$

$$\text{curvature tensors:}\quad \mathring{b}_{\alpha\beta} = -\mathring{\mathbf{a}}_\alpha\cdot\mathring{\mathbf{a}}_{3,\beta},\quad \mathring{b}_\alpha^\beta = \mathring{b}_{\alpha\rho}\mathring{a}^{\rho\beta}. \tag{4}$$

Herein, $\mathring{\mathbf{a}}_3$ denotes the unit normal vector of $\mathring{F}$ and the notation $(\ )_{,\alpha}$ partial derivatives with respect to θ^α. As usual, Greek indices represent the numbers 1, 2 and the Latin indices the numbers 1, 2, 3. The covariant derivatives with respect to the undeformed state $\mathring{F}$ will be denoted by $(\ )|_\alpha$.

The geometrical variables of the deformed middle surface F will be denoted, as are needed, without (°). Thus, $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$ are the base vectors associated with the deformed position P of the point $\mathring{P}$.

Let θ^3 be the distance of an arbitrary point $\mathring{P}^*$ of the undeformed shell from $\mathring{F}$, measured in the direction of $\mathring{\mathbf{a}}_3$. Thus, from the position vector (Fig. 1)

$$\mathring{\mathbf{r}}^* = \mathring{\mathbf{r}}+\theta^3\mathring{\mathbf{a}}_3 \tag{5}$$

we obtain for the base vectors $\mathring{\mathbf{a}}_i^*$ of the point $\mathring{P}^*$

$$\mathring{\mathbf{a}}_\alpha^* = \mathring{\mathbf{a}}_\alpha+\theta^3\mathbf{a}_{3,\alpha} = \mathring{\mu}_\alpha^\rho\mathring{\mathbf{a}}_\rho,\quad \mathring{\mathbf{a}}_3^* = \mathring{\mathbf{a}}_3 \tag{6}$$

where

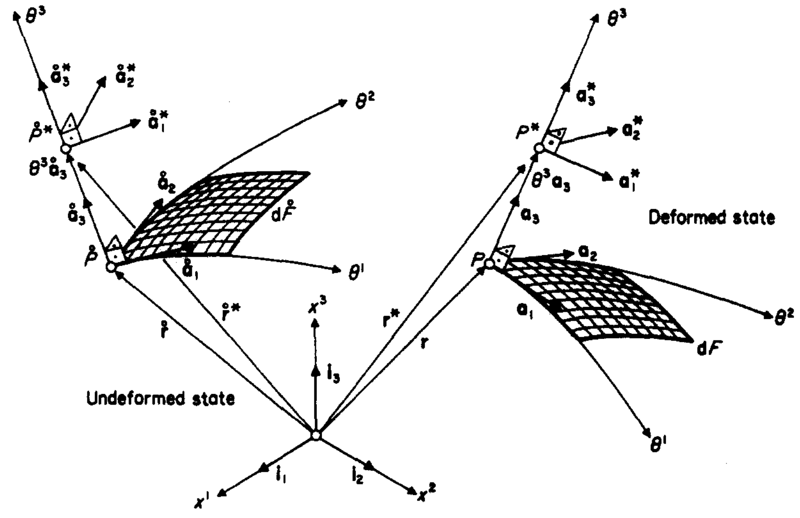

Fig. 1. The geometry of the shell continuum.

$$\mathring{\mu}^{\rho}_{\alpha} = \delta^{\rho}_{\alpha} - \theta^3 \mathring{b}^{\rho}_{\alpha}. \tag{7}$$

The notation ( )* will be used, also in the following, to characterize variables related to the shell continuum $\mathring{P}^*$.

Let P and P^* be the positions of $\mathring{P}$ and $\mathring{P}^*$, respectively, in the deformed state. If we now, according to eqn (13), assume that the distance between P and P^* has the value θ^3, then we can express the position vector $\mathbf{r}^*$ related to P^* by

$$\mathbf{r}^* = \mathbf{r} + \theta^3 \mathbf{a}_3 \tag{8}$$

from which it follows

$$\mathbf{a}^*_\alpha = \mathbf{a}_\alpha + \theta^3 \mathbf{a}_{3,\alpha}, \quad \mathbf{a}^*_3 = \mathbf{a}_3. \tag{9}$$

Here, $\mathbf{a}_3 = \mathbf{a}_3(\theta^\alpha)$ is a unit vector, which is not, however, perpendicular to the deformed middle surface F. Due to this fact, $\mathbf{a}^*_\alpha$ cannot be related, similar to $\mathring{\mathbf{a}}^*_\alpha$ eqns (6), to the curvature tensor of the corresponding middle surface. For later use we recall the definition [1, 7]

$$\mathring{a}^* = (\mathring{\mathbf{a}}^*_1 \times \mathring{\mathbf{a}}^*_2) \cdot \mathring{\mathbf{a}}^*_3 \tag{10}$$

so that the volume element $\mathrm{d}\mathring{V}$ of the shell continuum can be expressed as $\mathrm{d}\mathring{V} = \sqrt{\mathring{a}^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3$. Without the notation (°), eqn (10) is also valid for the deformed state.

Let us now assume that the middle surface $\mathring{F}$ is bounded by a smooth closed curve $\mathring{C}$, the line element of which is denoted by $\mathrm{d}\mathring{s}$. Along $\mathring{C}$, we introduce by the relations

$$\mathring{\mathbf{t}} = \frac{\mathrm{d}\mathring{\mathbf{r}}}{\mathrm{d}\mathring{s}} = \mathring{t}_\alpha \mathring{\mathbf{a}}^\alpha = \mathring{t}^\alpha \mathring{\mathbf{a}}_\alpha, \quad \mathring{\mathbf{u}} = \mathring{\mathbf{t}} \times \mathring{\mathbf{a}}_3 = \mathring{u}_\alpha \mathring{\mathbf{a}}^\alpha = \mathring{u}^\alpha \mathring{\mathbf{a}}_\alpha \tag{11}$$

the unit tangent vector $\mathring{\mathbf{t}}$ and the unit normal vector $\mathring{\mathbf{u}}$, both vectors lying in $\mathring{F}$. Vectors (11) are related by

$$\mathring{u}_\beta = \mathring{\varepsilon}_{\beta\alpha} \frac{\mathrm{d}\theta^\alpha}{\mathrm{d}\mathring{s}} = \mathring{\varepsilon}_{\beta\alpha} \mathring{t}^\alpha, \quad \mathring{t}^\alpha = \frac{\mathrm{d}\theta^\alpha}{\mathrm{d}\mathring{s}} = \mathring{\varepsilon}^{\beta\alpha} \mathring{u}_\beta \tag{12}$$

and define, together with $\mathring{\mathbf{a}}_3$, a right handed triad $(\mathring{\mathbf{u}}, \mathring{\mathbf{t}}, \mathring{\mathbf{a}}_3)$ which will be used later for the

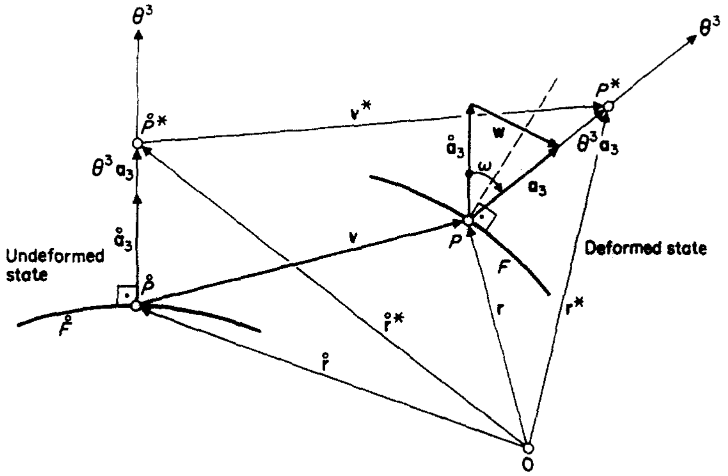

Fig. 2. The deformation state.

definition of physical boundary variables. In eqns (12), $\mathring{\varepsilon}_{\alpha\beta}$ is the permutation tensor associated with $\mathring{F}$.

3. STRAIN–DISPLACEMENT RELATIONS

For the definition of two-dimensional displacement variables we assume that points lying in the direction of the unit normal vector $\mathbf{\mathring{a}}_3$ are also after the deformation on a straight and that no changes of length occur in this direction. Thus, a point $\mathring{P}^*$ of the shell continuum takes, in the deformed state, the position P^*, according to Fig. 2 and the displacement vector $\mathbf{v}^*$ from $\mathring{P}^*$ to P^* can be expressed in the form

$$\mathbf{v}^* = \mathbf{r}^* - \mathbf{\mathring{r}}^* = \mathbf{r} - \mathbf{\mathring{r}} + \theta^3(\mathbf{a}_3 - \mathbf{\mathring{a}}_3)\mathbf{v} + \theta^3\mathbf{w} \tag{13}$$

where $\mathbf{a}_3$ is a unit vector which shows in the θ^3-direction of the deformed shell continuum.

The displacement vector of the middle surface $\mathbf{v}$ and the difference vector $\mathbf{w}$ will be expressed in terms of the base vectors $\mathbf{\mathring{a}}_i$ of the undeformed state $\mathring{F}$. Thus

$$\mathbf{v} = \mathbf{r} - \mathbf{\mathring{r}} = v_\alpha \mathbf{\mathring{a}}^\alpha + \mathring{v}_3 \mathbf{\mathring{a}}^3 = v^\alpha \mathbf{\mathring{a}}_\alpha + v^3 \mathbf{\mathring{a}}_3 \tag{14}$$

$$\mathbf{w} = \mathbf{a}_3 - \mathbf{\mathring{a}}_3 = w_\alpha \mathbf{\mathring{a}}^\alpha + \mathring{w}_3 \mathbf{\mathring{a}}^3 = w^\alpha \mathbf{\mathring{a}}_\alpha + w^3 \mathbf{\mathring{a}}_3. \tag{15}$$

Using the deformation gradients

$$\varphi_{\alpha\rho} = v_\rho|_\alpha - \mathring{b}_{\rho\alpha} v_3, \quad \varphi_{\alpha 3} = v_{3,\alpha} + \mathring{b}^\lambda_\alpha v_\lambda \tag{16}$$

$$\psi_{\alpha\rho} = w_\rho|_\alpha - \mathring{b}_{\rho\alpha} w_3, \quad \psi_{\alpha 3} = w_{3,\alpha} + \mathring{b}^\lambda_\alpha w_\lambda \tag{17}$$

which are related to the partial derivatives $\mathbf{v}_{,\alpha}$ and $\mathbf{w}_{,\alpha}$ by

$$\mathbf{v}_{,\alpha} = (v_\rho|_\alpha - \mathring{b}_{\rho\alpha} v_3)\mathbf{\mathring{a}}^\rho + (v_{3,\alpha} + \mathring{b}^\lambda_\alpha v_\lambda)\mathbf{\mathring{a}}^3 = \varphi_{\alpha\rho}\mathbf{\mathring{a}}^\rho + \varphi_{\alpha 3}\mathbf{\mathring{a}}^3 \tag{18}$$

$$\mathbf{w}_{,\alpha} = (w_\rho|_\alpha - \mathring{b}_{\rho\alpha} w_3)\mathbf{\mathring{a}}^\rho + (w_{3,\alpha} + \mathring{b}^\lambda_\alpha w_\lambda)\mathbf{\mathring{a}}^3 = \psi_{\alpha\rho}\mathbf{\mathring{a}}^\rho + \psi_{\alpha 3}\mathbf{\mathring{a}}^3 \tag{19}$$

we obtain from eqns (14) for the vectors $\mathbf{a}_\alpha$ of the deformed state F

$$\mathbf{a}_\alpha = \mathbf{\mathring{a}}_\alpha + \mathbf{v}_{,\alpha} = (\delta^\rho_\alpha + \varphi^\rho_{\alpha.})\mathbf{\mathring{a}}_\rho + \varphi_{\alpha 3}\mathbf{\mathring{a}}_3 \tag{20}$$

while the unit vector $\mathbf{a}_3$ is, according to eqns (15), given by

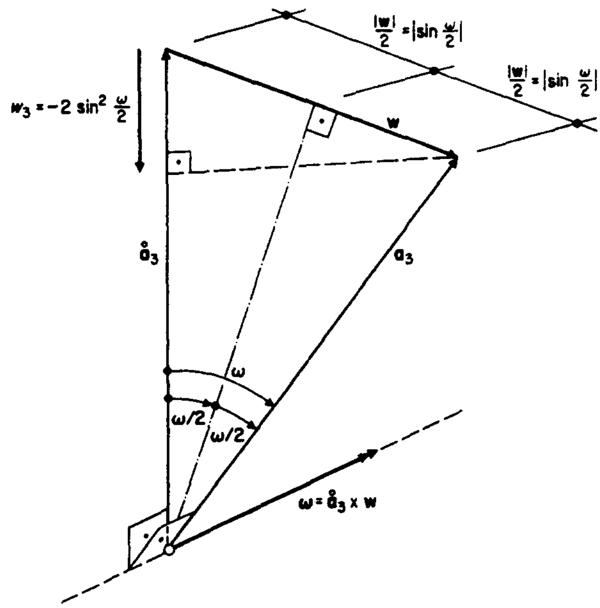

Fig. 3. The rotation vector.

$$\mathbf{a}_3 = \mathring{\mathbf{a}}_3 + \mathbf{w} = w^\alpha \mathring{\mathbf{a}}_\alpha + (1 + w_3)\mathring{\mathbf{a}}_3. \tag{21}$$

In view of our kinematic assumption that $\mathbf{a}_3 \cdot \mathbf{a}_3 = 1$, the normal component w_3 of the vector $\mathbf{w}$ may be given in terms of the tangential components w_α. Thus, using eqns (21) we have

$$\mathbf{a}_3 \cdot \mathbf{a}_3 = 1 \rightarrow w_3(2 + w_3) + w_\alpha w^\alpha = 0 \rightarrow w_3 = -1 \pm \sqrt{(1 - w_\alpha w^\alpha)} \tag{22}$$

where the negative sign in front of the square root has to be taken for the values $\pi/2 \leqslant \omega \leqslant 3\pi/2$ of the rotation angle ω defined in Fig. 3.

For later use, particularly for the formulation of geometrical boundary conditions, it is convenient to introduce the rotation vector $\boldsymbol{\omega}$, related to $\mathbf{w}$ by the vectorial product

$$\boldsymbol{\omega} = \omega_\alpha \mathring{\mathbf{a}}^\alpha = \omega^\alpha \mathring{\mathbf{a}}_\alpha = \mathring{\mathbf{a}}_3 \times \mathbf{w} \tag{23}$$

which, using eqns (15), gives

$$\omega_\alpha = \mathring{\varepsilon}_{\beta\alpha} w^\beta, \quad w_\alpha = \mathring{\varepsilon}_{\alpha\beta}\omega^\beta \tag{24}$$

where $\mathring{\varepsilon}_{\alpha\beta}$ is the permutation tensor of surface $\mathring{F}$. According to eqns (23) the vector $\boldsymbol{\omega}$ stands perpendicular to the plan defined by the rotation of $\mathring{\mathbf{a}}_3$ into $\mathbf{a}_3$. Furthermore, its magnitude is related to the angle of rotation ω by

$$|\boldsymbol{\omega}| = |\mathring{\mathbf{a}}_3| \left| \mathbf{w} \right| \left| \sin\left(\frac{\pi}{2} - \frac{\omega}{2}\right) \right| = 2\left| \sin\frac{\omega}{2}\cos\frac{\omega}{2} \right| = |\sin\omega| \tag{25}$$

as can easily be deduced from Fig. 3. Again from Fig. 3, we have

$$w_3 = -2\sin^2\frac{\omega}{2} = -\frac{1}{2}\mathbf{w}\cdot\mathbf{w} = -\frac{\sin^2\omega}{2\cos^2\frac{\omega}{2}} = -\frac{1}{2}\frac{\boldsymbol{\omega}\cdot\boldsymbol{\omega}}{\cos^2\frac{\omega}{2}} \tag{26}$$

so that

$$1+w_3 = 1-2\sin^2\frac{\omega}{2} = \cos\omega. \tag{27}$$

Substituting eqns (24) and (26) into eqns (15) the vector $\mathbf{w}$ can now be expressed solely in terms of $\boldsymbol{\omega}$ by the non-linear relations

$$\begin{aligned}\mathbf{w} = \mathbf{a}_3-\mathring{\mathbf{a}}_3 &= \boldsymbol{\omega}\times\mathring{\mathbf{a}}_3 - \frac{1}{2\cos^2\frac{\omega}{2}}(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathring{\mathbf{a}}_3 \\ &= \boldsymbol{\omega}\times\mathring{\mathbf{a}}_3 + \frac{1}{2\cos^2\frac{\omega}{2}}\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathring{\mathbf{a}}_3)\end{aligned} \tag{28}$$

showing the fact that $\boldsymbol{\omega}$ can be regarded as the rotation vector of $\mathring{\mathbf{a}}_3$[1, 13]. However, $\boldsymbol{\omega}$ must not be confused with the rotation vector of the base vector system $\mathring{\mathbf{a}}_i$ as is introduced by several authors[5, 14]. Soon, we shall discover the mechanical significance of $\boldsymbol{\omega}$ when we formulate the virtual work of physical stress couples.

For the definition of two-dimensional strain measures we use Green's strain tensor of a three-dimensional continuum defined by[1, 7, 12]

$$\gamma_{ij} = \tfrac{1}{2}(\mathbf{a}_i^*\cdot\mathbf{a}_j^* - \mathring{\mathbf{a}}_i^*\cdot\mathring{\mathbf{a}}_j^*) = \tfrac{1}{2}(\mathring{\mathbf{a}}_i^*\cdot\mathbf{v}_{,j}^* + \mathring{\mathbf{a}}_j^*\cdot\mathbf{v}_{,i}^* + \mathbf{v}_{,i}^*\cdot\mathbf{v}_{,j}^*). \tag{29}$$

In order to calculate the tangential components $\gamma_{\alpha\beta}$ we first introduce into this relation transformations (6) and (9). Then, using (7) and (18)–(21) and neglecting terms of second order in θ^3, we obtain

$$\gamma_{\alpha\beta} = \alpha_{\alpha\beta} + \theta^3\beta_{\alpha\beta} \tag{30}$$

where the abbreviations

$$\alpha_{\alpha\beta} = \alpha_{\beta\alpha} = \tfrac{1}{2}(\varphi_{\alpha\beta} + \varphi_{\beta\alpha} + \varphi_{\alpha\lambda}\varphi^{\lambda}_{\beta.} + \varphi_{\alpha 3}\varphi_{\beta 3}) \tag{31}$$

and

$$\begin{aligned}\beta_{\alpha\beta} = \beta_{\beta\alpha} = \tfrac{1}{2}[&w_\alpha|_\beta + w_\beta|_\alpha - \mathring{b}^\lambda_\alpha\varphi_{\beta\lambda} - \mathring{b}^\lambda_\beta\varphi_{\alpha\lambda} - 2\mathring{b}_{\alpha\beta}w_3 \\ &+ \varphi^\lambda_{\beta.}(w_\lambda|_\alpha - \mathring{b}_{\lambda\alpha}w_3) + \varphi^\lambda_{\alpha.}(w_\lambda|_\beta - \mathring{b}_{\lambda\beta}w_3) \\ &+ \varphi_{\beta 3}(w_{3,\alpha} + \mathring{b}^\lambda_\alpha w_\lambda) + \varphi_{\alpha 3}(w_{3,\beta} + \mathring{b}^\lambda_\beta w_\lambda)]\end{aligned} \tag{32}$$

denote the first $\alpha_{\alpha\beta}$ and the second strain tensor of the middle surface $\beta_{\alpha\beta}$. Using in addition the identity

$$(\mathbf{a}_3\cdot\mathbf{a}_3)_{,\alpha} = 0 \rightarrow \mathring{\mathbf{a}}_3\cdot\mathbf{w}_{,\alpha} - \mathring{b}^\rho_\alpha\mathring{\mathbf{a}}_\rho\cdot\mathbf{w} + \mathbf{w}_{,\alpha}\cdot\mathbf{w} = 0 \tag{33}$$

which has been found from eqns (21) and (22), we obtain similarly for $\gamma_{\alpha 3}$

$$\gamma_{\alpha 3} = \tfrac{1}{2}\gamma_\alpha = \tfrac{1}{2}[w_\rho(\delta^\rho_\alpha + \varphi^\rho_{\alpha.}) + \varphi_{\alpha 3}(1+w_3)]. \tag{34}$$

If we put the shear deformation γ_α equal to zero, then this relation degenerates to the well-known orthogonality condition $\mathbf{a}_\alpha\cdot\mathbf{a}_3 = 0$, characterizing the Kirchhoff–Love theories. Considering eqns (22), it can be finally shown that the remaining component γ_{33} of Green's tensor (29) vanishes identically. Thus, the state of deformation of the shell continuum is described entirely by the strain measures (31), (32) and (34) as

$$\gamma_{ij} = \left[\begin{array}{c|c} \gamma_{\alpha\beta} & \gamma_{\alpha 3} \\ \hline \gamma_{3\alpha} & \gamma_{33} \end{array}\right] = \left[\begin{array}{c|c} \alpha_{\alpha\beta}+\theta^3\beta_{\alpha\beta} & \frac{1}{2}\gamma_\alpha \\ \hline \frac{1}{2}\gamma_\alpha & 0 \end{array}\right]. \tag{35}$$

4. THE PRINCIPLE OF VIRTUAL WORK AND THE DEFINITION OF CONSISTENT FORCE VARIABLES

The two-dimensional internal forces will be called consistent if they are related, in the corresponding expression of virtual work, to the first variation of the strain variables introduced in eqns (31), (32) and (34). For their derivation, we shall use the three-dimensional expression of internal virtual work which can, in terms of the Cauchy stress tensor τ^{ij} and the Piola–Kirchhoff stress tensor of the second kind $s^{ij} = \sqrt{(a^*/\mathring{a}^*)}\tau^{ij}$, be given in the alternative forms

$$\delta^* A_i = -\iiint_V \tau^{ij}\,\delta\gamma_{ij}\sqrt{a^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3 = -\iiint_{\mathring{V}} \mathring{\mu} s^{ij}\,\delta\gamma_{ij}\sqrt{\mathring{a}}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3 \tag{36}$$

where

$$\mathring{\mu} = \sqrt{\left(\frac{\mathring{a}^*}{\mathring{a}}\right)} \tag{37}$$

is related to the undeformed shell continuum. Substituting eqn (35) into eqn (36) and remembering that the surface element $\mathrm{d}\mathring{F} = \sqrt{\mathring{a}}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2$ is independent of the parameter θ^3, we obtain the two-dimensional relation

$$\delta^* A_i = -\iint_{\mathring{F}} (\tilde{N}^{(\alpha\beta)}\,\delta\alpha_{\alpha\beta} + \tilde{Q}^\alpha\,\delta\gamma_\alpha + M^{(\alpha\beta)}\,\delta\beta_{\alpha\beta})\,\mathrm{d}\mathring{F} \tag{38}$$

where the consistent force variables defined as

$$\tilde{N}^{(\alpha\beta)} = \int_{-h/2}^{h/2} \mathring{\mu} s^{\alpha\beta}\,\mathrm{d}\theta^3,\quad \tilde{Q}^\alpha = \int_{-h/2}^{h/2} \mathring{\mu} s^{\alpha 3}\,\mathrm{d}\theta^3,\quad M^{(\alpha\beta)} = \int_{-h/2}^{h/2} \mathring{\mu} s^{\alpha\beta}\theta^3\,\mathrm{d}\theta^3 \tag{39}$$

are called the pseudo-stress resultant tensor, the pseudo-shear stress vector and the moment tensor, respectively. Because of the well-known symmetry $s^{\alpha\beta} = s^{\beta\alpha}$, both tensors $N^{(\alpha\beta)}$ and $M^{(\alpha\beta)}$ are also symmetrical which is indicated by putting the corresponding indices in round brackets. In eqns (39) h denotes the shell thickness.

In order to make the load variables accessible for practical applications it is convenient to introduce them directly on the two-dimensional middle surface element. Thus, we denote by $\mathbf{p}\,\mathrm{d}F$ and $\mathbf{c}\,\mathrm{d}F$ the load resultant and the load-moment resultant acting on the element $\mathrm{d}F$ of the deformed middle surface F. Now, we introduce, using the factor $\mathrm{d}F/\mathrm{d}\mathring{F} = \sqrt{(a/\mathring{a})}$ and transformations (20) and (21), the following load components:

$$\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{p} = P^\alpha\mathbf{a}_\alpha + P^3\mathbf{a}_3 = p^\alpha\mathring{\mathbf{a}}_\alpha + p^3\mathring{\mathbf{a}}_3,$$

$$\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{c} = C^\alpha\mathbf{a}_3\times\mathbf{a}_\alpha = (1+w_3)c^\alpha\mathring{\mathbf{a}}_3\times\mathring{\mathbf{a}}_\alpha + (\delta^\rho_\beta+\varphi^\rho_{\beta.})w^\lambda C^\beta\mathring{\mathbf{a}}_\lambda\times\mathring{\mathbf{a}}_\rho \tag{40}$$

which are, according to transformations (20) and (21), related by

$$p^\alpha = P^\beta(\delta^\alpha_\beta + \varphi^\alpha_{\beta.}) + P^3 w^\alpha, \quad p^3 = P^\beta \varphi_{\beta 3} + P^3(1+w_3),$$
$$c^\alpha = C^\beta\left(\delta^\alpha_\beta + \varphi^\alpha_{\beta.} - \frac{w^\alpha}{1+w_3}\varphi_{\beta 3}\right). \tag{41}$$

As in eqns (41), we shall also in the following denote the variables defined with respect of the deformed reference frame $\mathbf{a}_i$ by the upper case letters, preserving the lower case letters for those defined in the undeformed reference frame $\mathring{\mathbf{a}}_i$. The virtual work performed by the loads, eqns (40), along the independent virtual displacements $\delta\mathbf{v}$ and $\delta\boldsymbol{\omega}$ is, using eqns (14), (23) and (24), given by

$$\delta^* A_{aP} = \iint_{\mathring{F}} \left\{ \left(\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{p}\right)\cdot\delta\mathbf{v} + \left(\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{c}\right)\cdot\frac{\delta\boldsymbol{\omega}}{1+w_3}\right\} \mathrm{d}\mathring{F} = \iint_{\mathring{F}} (p^\alpha \delta v_\alpha + p^3 \delta v_3 + c^\alpha \delta w_\alpha)\, \mathrm{d}\mathring{F}. \tag{42}$$

The expression given above for the virtual work of the load couple vector $\mathbf{c}$ in terms of the rotation vector $\delta\boldsymbol{\omega}$ is at this stage an assumption, the validity of which, however, will be confirmed later by a three-dimensional derivation, eqns (68).

Finally, we have to formulate the virtual work of the inertia forces due to dynamic effects. The corresponding three-dimensional expression is, according to eqns (37), given by

$$\delta^* A_{aD} = -\iiint_{\mathring{V}} \rho\ddot{\mathbf{v}}^*\cdot\delta\mathbf{v}^*\sqrt{\mathring{a}^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3 = -\iiint_{\mathring{V}} \mathring{\rho}\mathring{\mu}\ddot{\mathbf{v}}^*\cdot\delta\mathbf{v}^*\sqrt{\mathring{a}}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3 \tag{43}$$

where $\mathring{\rho}$ is the mass density of the undeformed shell continuum and dots ($\dot{\ }$) denote derivatives with respect to time. Introducing eqns (13) into eqns (43) and using the approximation $\mathring{\mu} \simeq 1$, we obtain after integration with respect to θ^3

$$\delta^* A_{aD} = -\mathring{\rho}\iint_{\mathring{F}}\left(h\ddot{\mathbf{v}}\cdot\delta\mathbf{v} + \frac{h^3}{12}\ddot{\mathbf{w}}\cdot\delta\mathbf{w}\right)\mathrm{d}\mathring{F}. \tag{44}$$

If we now substitute $\mathbf{v}$ and $\mathbf{w}$ from eqns (14) and (15) and use, according to eqns (22), the relation

$$\delta w_3(1+w_3) + w^\alpha \delta w_\alpha = 0 \rightarrow \delta w_3 = -\frac{w^\alpha}{1+w_3}\delta w_\alpha = -\frac{w^\alpha}{\cos\omega}\delta w_\alpha \tag{45}$$

for the elimination of the dependent variation δw_3, then expression (44) reduces to

$$\delta^* A_{aD} = -\mathring{\rho}\iint_{\mathring{F}}\left\{h(\ddot{v}^\beta\delta v_\beta + \ddot{v}^3\delta v_3) + \frac{h^3}{12}\left(\ddot{w}^\beta - \frac{w^\beta}{1+w_3}\ddot{w}_3\right)\delta w_\beta\right\}\mathrm{d}\mathring{F}. \tag{46}$$

Since the force variables which can be prescribed along the boundary are still not defined we now consider a boundary value problem with prescribed boundary displacements ($\delta v_i = \delta w_\alpha = 0$). In this case the virtual work of the forces acting upon the boundary curve of the shell vanishes indentically and the principle of virtual work can, using eqns (38), (42) and (46), be expressed in the form

$$\delta^* A = \delta^* A_a + \delta^* A_i = \iint_{\mathring{F}} (p^\beta \delta v_\beta + p^3 \delta v_3 + c^\beta \delta w_\beta)\, d\mathring{F}$$
$$- \mathring{\rho} \iint_{\mathring{F}} \left\{ h(\ddot{v}^\beta \delta v_\beta + \ddot{v}^3 \delta v_3) + \frac{h^3}{12}\left(\ddot{w}^\beta - \frac{w^\beta}{1+w_3}\ddot{w}_3\right)\delta w_\beta \right\} d\mathring{F}$$
$$- \iint_{\mathring{F}} (\tilde{N}^{(\alpha\beta)}\, \delta\alpha_{\alpha\beta} + \tilde{Q}^\alpha \delta\gamma_\alpha + M^{(\alpha\beta)} \delta\beta_{\alpha\beta})\, d\mathring{F} = 0. \quad (47)$$

5. THE EQUATIONS OF MOTION AND THE DEFINITION OF TENSORIAL BOUNDARY FORCES

The corresponding relations will be derived here from the principle of virtual work, eqns (47), using the well-known rules of the calculus of variation. This procedure makes it possible to derive relations which are in the same order of accuracy as the initial kinematic relations (31), (32) and (34) and which can be, therefore, regarded as a consistent formulation.

For the derivation we first express in eqns (47) the strains $\alpha_{\alpha\beta}$, $\beta_{\alpha\beta}$ and γ_α by eqns (31), (32) and (34). Then, we substitute the dependent variation δw_3 from constraint (45) in order to transform in the following all the terms connected with the covariant derivatives $\delta v_{i|\beta}$ and $\delta w_{\alpha|\beta}$ by means of Green's theorem, for instance, according to

$$\iint_{\mathring{F}} M^{(\alpha\beta)}\varphi_{\beta 3}\left(\frac{w^\lambda}{1+w_3}\delta w_\lambda\right)\Big|_\alpha\, d\mathring{F} = \oint_{\mathring{C}} M^{(\alpha\beta)}\varphi_{\beta 3}\frac{w^\lambda}{1+w_3}\mathring{u}_\alpha \delta w_\lambda\, d\mathring{F}$$
$$- \iint_{\mathring{F}} (M^{(\alpha\beta)}\varphi_{\beta 3})|_\alpha \frac{w^\lambda}{1+w_3}\delta w_\lambda\, d\mathring{F} \quad (48)$$

where $\mathring{\mathbf{u}} = \mathring{u}_\alpha \mathring{\mathbf{a}}^\alpha$ is the unit normal of the boundary curve $\mathring{C}$. This procedure leads to the following equations of motion:

$$n^{\alpha\beta}|_\alpha - \mathring{b}^\beta_\alpha q^\alpha + p^\beta - \mathring{\rho} h \ddot{v}^\beta = 0, \quad \mathring{b}_{\alpha\beta} n^{\alpha\beta} + q^\alpha|_\alpha + p^3 - \mathring{\rho} h \ddot{v}_3 = 0,$$
$$m^{\alpha\beta}|_\alpha - q^{*\beta} + c^\beta - \mathring{\rho}\frac{h^3}{12}\left(\ddot{w}^\beta - \frac{w^\beta}{1+w_3}\ddot{w}_3\right) = 0 \quad (49)$$

where the first three abbreviations

$$n^{\alpha\beta} = \tilde{N}^{(\alpha\rho)}(\delta^\beta_\rho + \varphi^\beta_{\rho.}) - M^{(\alpha\rho)}(\mathring{b}^\beta_\rho - \psi^\beta_{\rho.}) + \tilde{Q}^\alpha w^\beta,$$
$$q^\alpha = \tilde{Q}^\alpha(1+w_3) + \tilde{N}^{(\alpha\rho)}\varphi_{\rho 3} + M^{(\alpha\rho)}\psi_{\rho 3},$$
$$m^{\alpha\beta} = M^{(\alpha\rho)}\left(\delta^\beta_\rho + \varphi^\beta_{\rho.} - \frac{w^\beta}{1+w_3}\varphi_{\rho 3}\right), \quad (50)$$
$$q^{*\beta} = \tilde{Q}^\rho\left(\delta^\beta_\rho + \varphi^\beta_{\rho.} - \frac{w^\beta}{1+w_3}\varphi_{\rho 3}\right)$$
$$+ M^{(\alpha\rho)}\left\{\mathring{b}_{\alpha\lambda}(\delta^\lambda_\rho + \varphi^\lambda_{\rho.})\frac{w^\beta}{1+w_3} + \left[\mathring{b}^\beta_\alpha - \left(\frac{w^\beta}{1+w_3}\right)\Big|_\alpha\right]\varphi_{\rho 3}\right\}$$

are, as will be shown in eqns (54) and (58), force variables defined with respect of the undeformed reference frame $\mathring{\mathbf{a}}_i$. The last variable $q^{*\beta}$ is, however, not interpretable in this sense.

Furthermore, we deduce from the line integral

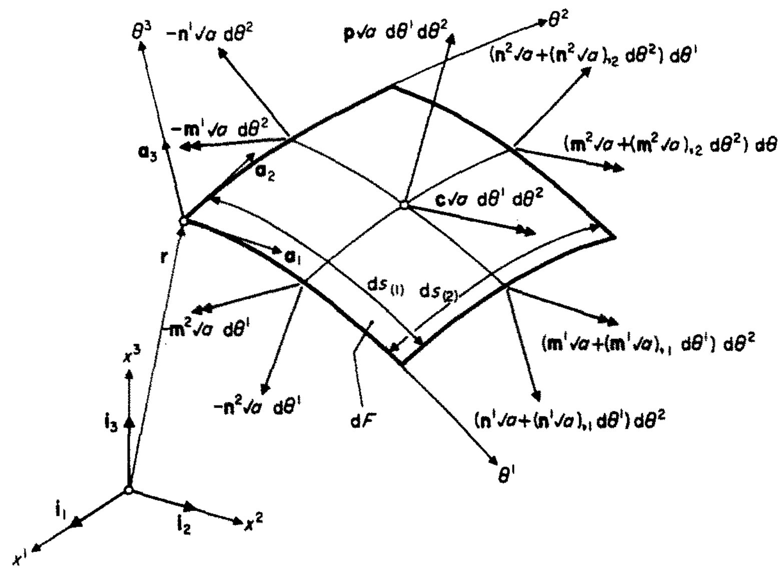

Fig. 4. The equilibrium of the middle surface element.

$$\delta^* A_{aC} = \oint_{\mathring{C}} (n^{\alpha\beta}\delta v_\beta + q^\alpha \delta v_3 + m^{\alpha\beta}\delta w_\beta)\mathring{u}_\alpha \, \mathrm{d}\mathring{s} \tag{51}$$

obtained also by the same procedure, the definition of the tensorial force variables which can be prescribed along the boundary of the shell $\mathring{C}$

$$n^\beta = n^{\alpha\beta}\mathring{u}_\alpha, \quad n^3 = q^\alpha \mathring{u}_\alpha, \quad m^\beta = m^{\alpha\beta}\mathring{u}_\alpha. \tag{52}$$

Integral (51) expresses the virtual work of the boundary forces and vanishes identically, if the boundary displacements v_i and w_α are, as are supposed above for convenience, all prescribed.

6. THE PHYSICAL INTERPRETATION OF THE FORCE VARIABLES

All the force variables have been introduced until now using a variational procedure. In the following we have to relate them with physical variables interpretable on a two-dimensional shell element in order to make the theory accessible for practical applications. Moreover, we are interested to find out if the equations of motions, eqns (49), can also be obtained by a two-dimensional derivation. Especially in non-linear theories, it is rather difficult to obtain equations in full coincidence using two different approaches.

Now, we consider, as illustrated in Fig. 4, an element of the deformed middle surface. Let $\mathbf{n}^\alpha\sqrt{a}\,\mathrm{d}\theta^\beta$ be the stress resultant vector acting on an element of the coordinate line $\theta^\alpha = \text{const}$, having the length $\mathrm{d}s_{\langle\beta\rangle} = \sqrt{a_{\beta\beta}}\,\mathrm{d}\theta^\beta$ $(\alpha \neq \beta)$. The stress couple resultant acting on the same element will be denoted by $\mathbf{m}^\alpha\sqrt{a}\,\mathrm{d}\theta^\beta$. By the well-known integration procedure of the stresses s^{ij} with respect of the parameter θ^3 it can be shown that

$$\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{n}^\alpha = \int_{-h/2}^{h/2} \mathring{\mu} s^{\alpha i}\mathbf{a}_i^* \, \mathrm{d}\theta^3, \quad \sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{m}^\alpha = \int_{-h/2}^{h/2} \mathring{\mu}(\mathbf{a}_3^* \times \mathbf{a}_\beta^*) s^{\alpha\beta}\theta^3 \, \mathrm{d}\theta^3. \tag{53}$$

With the help of eqns (9), (19), (20) and (21) and in view of definitions (39) and (50), the first relation reduces to

$$\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{n}^\alpha = n^{\alpha\beta}\mathring{\mathbf{a}}_\beta + q^\alpha \mathring{\mathbf{a}}_3 \tag{54}$$

showing the geometrical meaning of the abbreviations introduced in definitions (50). Introducing eqns (20) and (21) into the decomposition

$$\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{n}^\alpha = N^{\alpha\beta}\mathbf{a}_\beta + Q^\alpha \mathbf{a}_3 \tag{55}$$

with respect of the deformed reference frame $\mathbf{a}_i$, we obtain furthermore by comparison with eqn (54)

$$n^{\alpha\beta} = (\delta^\beta_\rho + \varphi^\beta_{\rho.})N^{\alpha\rho} + w^\beta Q^\alpha, \quad q^\alpha = (1+w_3)Q^\alpha + \varphi_{\beta 3}N^{\alpha\beta} \tag{56}$$

and hence, in view of definitions (50)

$$\begin{aligned} (\delta^\beta_\rho + \varphi^\beta_{\rho.})(\tilde{N}^{(\alpha\rho)} - N^{\alpha\rho}) &= w^\beta(Q^\alpha - \tilde{Q}^\alpha) + (\mathring{b}^\beta_\rho - \psi^\beta_{\rho.})M^{(\alpha\rho)}, \\ (1+w_3)(\tilde{Q}^\alpha - Q^\alpha) &= \varphi_{\beta 3}(N^{\alpha\beta} - \tilde{N}^{(\alpha\beta)}) - \psi_{\beta 3}M^{(\alpha\beta)}. \end{aligned} \tag{57}$$

According to these non-linear relations, the variationally defined forces $\tilde{N}^{(\alpha\beta)}$ and $\tilde{Q}^\alpha$, eqns (39) can be transformed into the geometrically defined ones $N^{\alpha\beta}$ and Q^α, eqn (55).

Considering in addition the identity $\mathbf{a}_3 \cdot \mathbf{a}_{3,\alpha} = 0$, eqn $(53)_2$ can, by a similar procedure, be transformed into

$$\begin{aligned} \sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{m}^\alpha &= M^{(\alpha\beta)}\mathbf{a}_3 \times \mathbf{a}_\beta \\ &= (1+w_3)m^{\alpha\beta}\mathring{\mathbf{a}}_3 \times \mathring{\mathbf{a}}_\beta + w^\lambda(\delta^\beta_\rho + \varphi^\beta_{\rho.})M^{(\alpha\rho)}\mathring{\mathbf{a}}_\lambda \times \mathring{\mathbf{a}}_\beta. \end{aligned} \tag{58}$$

This shows the physical meaning of the moment tensor $M^{(\alpha\beta)}$ as components of the stress couple vector in terms of the base vectors $\mathbf{a}_3 \times \mathbf{a}_\beta = \varepsilon_{\beta\rho}\mathbf{a}^\rho$ of the deformed state F. In view of eqn (55) a similar interpretation can also be given for the variables $N^{\alpha\beta}$ and Q^α while $\tilde{N}^{(\alpha\beta)}$ and $\tilde{Q}^\alpha$ are in this sense not directly interpretable and called therefore pseudo-forces. Remembering that $M^{(\alpha\beta)}$ and $\mathbf{m}^\alpha$, eqns (53), were first introduced independently we can deduce from eqns (58) that the kinematic assumption (13) used in eqn (38) for the definition of $M^{(\alpha\beta)}$ implies a stress couple vector $\mathbf{m}^\alpha$ which must be perpendicular to $\mathbf{a}_3$.

Finally, it can be shown that

$$\begin{bmatrix} N^{\langle\alpha\beta\rangle} \\ M^{\langle\alpha\beta\rangle} \end{bmatrix} = \sqrt{\left(\frac{\mathring{a}}{a}\right)}\sqrt{\left(\frac{a_{\beta\beta}}{a^{\alpha\alpha}}\right)}\begin{bmatrix} N^{\alpha\beta} \\ M^{(\alpha\beta)} \end{bmatrix}, \quad Q^{\langle\alpha\rangle} = \sqrt{\left(\frac{\mathring{a}}{a}\right)}\frac{1}{\sqrt{a^{\alpha\alpha}}}Q^\alpha \tag{59}$$

where $N^{\langle\alpha\beta\rangle}$, $M^{\langle\alpha\beta\rangle}$ and $Q^{\langle\alpha\rangle}$ are physical force variables, having the same direction as the corresponding tensorial components $N^{\alpha\beta}$, $M^{(\alpha\beta)}$ and Q^α, referring however to the unit length of the coordinate lines θ^α = const of the deformed middle surface F. For an arbitrary large displacement it is not possible to relate the physical components (59) directly to the pseudo-forces $\tilde{N}^{(\alpha\beta)}$ and $\tilde{Q}^\alpha$, which have first to be transformed into $N^{\alpha\beta}$ and Q^α according to the non-linear transformations (57).

According to Fig. 4 the vectorial equilibrium equations of the shell element are given by

$$\left(\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{n}^{\alpha}\right)\Big|_{\alpha}+\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{p}=0,\quad \left(\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{m}^{\alpha}\right)\Big|_{\alpha}+\mathbf{a}_{\alpha}\times\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{n}^{\alpha}+\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{c}=0 \tag{60}$$

which, using eqns (20), (21), (40), (54) and (58), can be transformed into the same component equations as are established in eqns (49) by the variational method. This can be regarded as a significant advantage of the equations presented here.

In order to define physical boundary forces, we introduce along the deformed boundary curve C the stress resultant $\mathbf{n}$ and the stress couple $\mathbf{m}$, both vectors referring to the unit length of the same curve. Formulating the equilibrium conditions of a shell subjected to the loads $\mathbf{p}$ and $\mathbf{c}$ and, along the boundary curve C, to $\mathbf{n}$ and $\mathbf{m}$, it can be shown with the help of Green's theorem and the equilibrium equations (60) that

$$\frac{\mathrm{d}s}{\mathrm{d}\mathring{s}}\mathbf{n}=\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{n}^{\alpha}\mathring{u}_{\alpha},\quad \frac{\mathrm{d}s}{\mathrm{d}\mathring{s}}\mathbf{m}=\sqrt{\left(\frac{a}{\mathring{a}}\right)}\mathbf{m}^{\alpha}\mathring{u}_{\alpha} \tag{61}$$

$\mathring{\mathbf{u}}=\mathring{u}_{\alpha}\mathring{\mathbf{a}}^{\alpha}$ being the normal vector of the undeformed curve $\mathring{C}$. With the help of the vectors $\mathring{\mathbf{u}}$, $\mathring{\mathbf{t}}$ (11) and $\mathring{\mathbf{a}}_3$ we now introduce the following physical force variables along the boundary $\mathring{C}$:

$$\frac{\mathrm{d}s}{\mathrm{d}\mathring{s}}\mathbf{n}=n_t\mathring{\mathbf{t}}+n_u\mathring{\mathbf{u}}+n_3\mathring{\mathbf{a}}_3,\quad \frac{\mathrm{d}s}{\mathrm{d}\mathring{s}}\mathbf{m}=m_t\mathring{\mathbf{t}}+m_u\mathring{\mathbf{u}}+m_3\mathring{\mathbf{a}}_3 \tag{62}$$

which are, due to the scaling factor $\mathrm{d}s/\mathrm{d}\mathring{s}$ used by the decomposition, force variables per unit length of the undeformed curve $\mathring{C}$.

Physical boundary displacements can be defined similarly by

$$\mathbf{v}=v_t\mathring{\mathbf{t}}+v_u\mathring{\mathbf{u}}+v_3\mathring{\mathbf{a}}_3,\quad \boldsymbol{\omega}=\omega_t\mathring{\mathbf{t}}+\omega_u\mathring{\mathbf{u}} \tag{63}$$

where, from eqns (11), (12), (14), (23) and (24)

$$v_t=v_{\alpha}\mathring{t}^{\alpha},\quad v_u=v_{\alpha}\mathring{u}^{\alpha},\quad v_3=v^3,\quad \omega_t=\omega_{\alpha}\mathring{t}^{\alpha}=w_{\alpha}\mathring{u}^{\alpha},\quad \omega_u=\omega_{\alpha}\mathring{u}^{\alpha}=-w_{\alpha}\mathring{t}^{\alpha}. \tag{64}$$

Since the vector $\mathbf{m}$ is, according to eqns (58) and (61), perpendicular to $\mathbf{a}_3$ ($\mathbf{m}\cdot\mathbf{a}_3=0$) the component m_3 (62) is a dependent variable which can be expressed in terms of m_t and m_u. Thus, the scalar product of eqn (63), with $\mathbf{a}_3$ from eqns (21) yields

$$m_3=-\frac{w^{\alpha}}{1+w_3}(m_t\mathring{t}_{\alpha}+m_u\mathring{u}_{\alpha}). \tag{65}$$

For the definition of the physical boundary forces which can be prescribed along the boundary of the shell we now refer to the virtual work δ^*A_{aC} (51) which we have to transform first into a vectorial form. According to eqns (20), (21), (28) and (50) we can write

$$\begin{aligned}M^{(\alpha\beta)}(\mathbf{a}_3\times\mathbf{a}_{\beta})\cdot\delta\boldsymbol{\omega}&=(1+w_3)m^{\alpha\beta}(\mathring{\mathbf{a}}_3\times\mathring{\mathbf{a}}_{\beta})\cdot\delta\boldsymbol{\omega}=(1+w_3)m^{\alpha\beta}\mathring{\mathbf{a}}_{\beta}\cdot\delta\mathbf{w}\\&=(1+w_3)m^{\alpha\beta}\delta w_{\beta}.\end{aligned} \tag{66}$$

Thus, considering also eqns (14), (54) and (61), it can be verified that expression (51) is identical with the following one:

$$\delta^* A_{aC} = \oint_{\mathring{C}} \left[\left(\sqrt{\left(\frac{a}{\mathring{a}}\right)} \mathbf{n}^\alpha \mathring{u}_\alpha \right) \cdot \delta \mathbf{v} + \frac{1}{1+w_3} \left(\sqrt{\left(\frac{a}{\mathring{a}}\right)} \mathbf{m}^\alpha \mathring{u}_\alpha \right) \cdot \delta \boldsymbol{\omega} \right] \mathrm{d}\mathring{s}$$
$$= \oint_{\mathring{C}} \left[\left(\frac{\mathrm{d}s}{\mathrm{d}\mathring{s}} \mathbf{n} \right) \cdot \delta \mathbf{v} + \frac{1}{1+w_3} \left(\frac{\mathrm{d}s}{\mathrm{d}\mathring{s}} \mathbf{m} \right) \cdot \delta \boldsymbol{\omega} \right] \mathrm{d}\mathring{s} \tag{67}$$

which in turn by virtue of eqns (62) and (63) reduces to

$$\delta^* A_{aC} = \oint_{\mathring{C}} \left(n_t \delta v_t + n_u \delta v_u + n_3 \delta v_3 + \frac{m_t}{1+w_3} \delta \omega_t + \frac{m_u}{1+w_3} \delta \omega_u \right) \mathrm{d}\mathring{s}. \tag{68}$$

The vectorial expression (67) shows that $\delta\boldsymbol{\omega}$ is the variable needed to express the virtual work of the physical stress couple $(\mathrm{d}s/\mathrm{d}\mathring{s})\mathbf{m}$, which justifies its definition in this theory. From eqn (68) we seen that $n_t, n_u, \ldots$ are the physical force variables to be prescribed in this theory as they are connected with the variation of the independent boundary displacements.

If we now substitute eqn $(24)_2$ into eqn (51) in order to replace δv_α and $\delta \omega_\alpha$ by

$$\delta v_\alpha = \delta v_t \mathring{t}_\alpha + \delta v_u \mathring{u}_\alpha, \quad \delta \omega_\alpha = \delta \omega_t \mathring{t}_\alpha + \delta \omega_u \mathring{u}_\alpha \tag{69}$$

we obtain in comparison with eqn (68) the following transformations:

$$\begin{aligned} n_t &= n^{\alpha\beta} \mathring{u}_\alpha \mathring{t}_\beta, & n_u &= n^{\alpha\beta} \mathring{u}_\alpha \mathring{u}_\beta, \quad n_3 = q^\alpha \mathring{u}_\alpha, \\ m_t &= (1+w_3) m^{\alpha\beta} \mathring{u}_\alpha \mathring{u}_\beta, & m_u &= -(1+w_3) m^{\alpha\beta} \mathring{u}_\alpha \mathring{t}_\beta \end{aligned} \tag{70}$$

for the physical force variables to be prescribed along the boundary curve $\mathring{C}$.

Considering the virtual work of boundary forces given in eqn (68) and the following boundary conditions:

$$\mathring{C}_r: \ \mathbf{v} = \mathbf{v}^\circ, \quad \boldsymbol{\omega} = \boldsymbol{\omega}^\circ, \qquad \mathring{C}_t: \ \mathbf{n} = \mathbf{n}^\circ, \quad \mathbf{m} = \mathbf{m}^\circ \tag{71}$$

along the parts $\mathring{C}_t$ and $\mathring{C}_r$ of the boundary curve $\mathring{C}$, the principle of virtual work takes finally the form

$$\delta^* A = \delta^* A_a + \delta^* A_i = \iint_{\mathring{F}} (p^\beta \delta v_\beta + p^3 \delta v_3 + c^\beta \delta w_\beta) \, \mathrm{d}\mathring{F} + \int_{\mathring{C}_t} \left(n_t^\circ \delta v_t + n_u^\circ \delta v_u + n_3^\circ \delta v_3 \right.$$
$$\left. + \frac{m_t^\circ}{1+w_3} \delta \omega_t + \frac{m_u^\circ}{1+w_3} \delta \omega_u \right) \mathrm{d}\mathring{s} - \mathring{\rho} \iint_{\mathring{F}} \left\{ h(\ddot{v}^\beta \delta v_\beta + \ddot{v}^3 \delta v_3) \right.$$
$$\left. + \frac{h^3}{12} \left(\ddot{w}^\beta - \frac{w^\beta}{1+w_3} \ddot{w}_3 \right) \delta w_\beta \right\} \mathrm{d}\mathring{F} - \iint_{\mathring{F}} (\tilde{N}^{(\alpha\beta)} \delta \alpha_{\alpha\beta} + \tilde{Q}^\alpha \delta \gamma_\alpha + M^{(\alpha\beta)} \delta \beta_{\alpha\beta}) \, \mathrm{d}\mathring{F} = 0 \tag{72}$$

where the notation $(\ )^\circ$ characterize the given boundary values.

7. CONSTITUTIVE EQUATIONS

For hyperelastic shells the constitutive equations can be expressed in general form as

$$\tilde{N}^{(\alpha\beta)} = \frac{1}{2}\left(\frac{\partial \pi_i}{\partial \alpha_{\alpha\beta}} + \frac{\partial \pi_i}{\partial \alpha_{\beta\alpha}}\right), \quad M^{(\alpha\beta)} = \frac{1}{2}\left(\frac{\partial \pi_i}{\partial \beta_{\alpha\beta}} + \frac{\partial \pi_i}{\partial \beta_{\beta\alpha}}\right), \quad \tilde{Q}^{\alpha} = \frac{\partial \pi_i}{\partial \gamma_\alpha} \tag{73}$$

where π_i is the specific energy density per unit area of the undeformed middle surface. Now, we restrict our attention to a Hookean material the physical properties of which are characterized by Young's modulus E and Poisson's ratio ν. If we furthermore assume that the strain measures $\alpha_{\alpha\beta}$ and γ_α are of negligible order of magnitude in comparison with unity then the following expression can be derived for π_i according to the well-known procedure of the shell theory[1]

$$\pi_i = \tfrac{1}{2}(DH^{\alpha\beta\lambda\mu}\alpha_{\alpha\beta}\alpha_{\lambda\mu} + BH^{\alpha\beta\lambda\mu}\beta_{\alpha\beta}\beta_{\lambda\mu} + Gh\mathring{a}^{\alpha\lambda}\gamma_\alpha\gamma_\lambda) \tag{74}$$

where

$$H^{\alpha\beta\lambda\mu} = \frac{1-\nu}{2}\left(\mathring{a}^{\alpha\lambda}\mathring{a}^{\beta\mu} + \mathring{a}^{\alpha\mu}\mathring{a}^{\beta\lambda} + \frac{2\nu}{1-\nu}\mathring{a}^{\alpha\beta}\mathring{a}^{\lambda\mu}\right) \tag{75}$$

and

$$D = \frac{Eh}{1-\nu^2}, \quad B = \frac{Eh^3}{12(1-\nu^2)}, \quad Gh = \frac{Eh}{2(1+\nu)}. \tag{76}$$

Substituting eqn (74) into eqns (73) yields the relations

$$\tilde{N}^{(\alpha\beta)} = DH^{\alpha\beta\lambda\mu}\alpha_{\lambda\mu}, \quad M^{(\alpha\beta)} = BH^{\alpha\beta\lambda\mu}\beta_{\lambda\mu}, \quad \tilde{Q}^{\alpha} = Gh\mathring{a}^{\alpha\lambda}\gamma_\lambda \tag{77}$$

which are formally identical with those of the linear theory[1]. In view of eqn (74) the internal virtual work given in eqns (72) can now be expressed as a complete variation of the elastic potential π_i

$$\delta^* A_i = -\delta\Pi_i = -\delta \int\int_{\mathring{F}} \pi_i \, d\mathring{F}. \tag{78}$$

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